Section 3.2

nth Order Linear DE:

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \dots + P_{n-1}(x)y' + P_n(x)y = F(x)$$

assume unless otherwise noted that
① $P_i(x)$ and $F(x)$ are continuous on open I and
② $P_0(x) \neq 0$ @ each point on I

divide by $P_0(x)$ and do similiar substitution as in previous §

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

non-homogeneous

then associated homogeneous linear eq'n is:

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y = 0$$

thm: let $y_1, y_2, \dots, y_n$ be sol'ns on I then LC ← linear combination

$y = c_1y_1 + c_2y_2 + \dots + c_ny_n$ is also a sol'n
where $c_1, c_2, \dots, c_n$ are constants

given $y''' + 3y'' + 4y' + 12y = 0$ on $\mathbb{R}$
verify $y_1 = e^{-3x}$, $y_2 = \cos 2x$, $y_3 = \sin 2x$ are all sol'ns
try on your later

by thm above, gen soln is $y(x) = c_1e^{-3x} + c_2\cos 2x + c_3\sin 2x$

recall: a 2^{nd} order linear DE needs 2 initial conditions to find a particular solution

similarly, an nth order linear DE needs n initial conditions

Linear Independence vs. Dependence

recall: 2 sol'ns are linearly dependent if 1 is a constant multiple of other recall: 2 sol'ns are linearly dependent if 1 is a constant multiple of other

ex. given $f_1 = kf_2$ or $f_2 = kf_1$

$$\underset{c_1}{(1)}f_1 - \underset{c_2}{k}f_2 = 0 \qquad \underset{c_1}{(1)}f_2 - \underset{c_2}{k}f_1 = 0$$

c_1, c_2 in both scenarios are non-zero

s.t. $c_1 f_1 + c_2 f_2 = 0$

<u>Linear Dependence Def'n</u>

$f_1, f_2, \ldots, f_n$ are linearly dependent when there exists non-zero $c_1, c_2, \ldots, c_n$ s.t. $c_1 f_1 + c_2 f_2 + \ldots + c_n f_n = 0$

ex. $f_1 = \sin 2x$ $f_2 = \sin x \cos x$ $f_3 = e^x$ are linearly dependent

beccause $1f_1 - 2f_2 + \cancel{0f_3}^{\to 0} = 0$

$$1f_1 = 2f_2$$

$$\sin 2x = 2\sin x \cos x$$

↳ constant multiplo ⇒ linear dependence

conversely, $f_1, f_2, \ldots, f_n$ are linearly <u>in</u>dependent when the identity $c_1 f_1 + c_2 f_2 + \ldots + c_n f_n = 0$ <u>only</u> holds when $c_1 = c_2 = \ldots = c_n = 0$] called trivial case

$c_1 = 1 \neq 0$, $c_2 = -2 \neq 0$ ← non-trivial case

3x3 determinant

cell $= a_{ij}$ (row i, column j)

ex $\begin{bmatrix} \cdot & \cdot & \cdot \\ a_{21} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

reduce to three 2x2 matrices

$$\boxed{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc}$$

$$\det A = +a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

ex. use Wronskian to show linear independence $y_1 = e^{-3x}$

$y_2 = \cos 2x$

$y_3 = \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y'' \end{vmatrix}$$

$$W = \begin{vmatrix} e^{-3x} & \cos 2x & \sin 2x \\ -3e^{-3x} & -2\sin 2x & 2\cos 2x \\ 9e^{-3x} & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$= e^{-3x}\begin{vmatrix} -2\sin 2x & 2\cos 2x \\ -4\cos 2x & -4\sin 2x \end{vmatrix} - (-3e^{-3x})\begin{vmatrix} \cos 2x & \sin 2x \\ -4\cos 2x & -4\sin 2x \end{vmatrix} + 9e^{-3x}\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= e^{-3x}\underbrace{(8\sin^2 2x + 8\cos^2 2x)}_{8(\sin^2 \sim + \cos^2 \sim)} + \cancel{3e^{-3x}(-4\sin 2x\cos 2x + 4\sin 2x\cos 2x)} + 9e^{-3x}(2\cos^2 2x + 2\sin^2 2x)$$

$$= 8e^{-3x} + 18e^{-3x}$$

$$= 26\underbrace{e^{-3x}}_{\text{never zero}} \neq 0$$

$\therefore$ y_1, y_2, y_3 are linearly independent

ex. show $y_1 = x$, $y_2 = x\ln x$, $y_3 = x^2$ are linearly independent sol'ns on $x > 0$

$$W = \begin{vmatrix} x & x\ln x & x^2 \\ 1 & \ln x + 1 & 2x \\ 0 & \frac{1}{x} & 2 \end{vmatrix} = x\begin{vmatrix} \ln x + 1 & 2x \\ \frac{1}{x} & 2 \end{vmatrix} - \begin{vmatrix} x\ln x & x^2 \\ \frac{1}{x} & 2 \end{vmatrix} + 0$$

$$= x(2\ln x + \cancel{2 - 2}) - (2x\ln x - x)$$

$$= \cancel{2x\ln x} - \cancel{2x\ln x} + x$$

$$= x$$

since $W \neq 0$ for $x > 0$, then y_1, y_2, y_3 are linearly independent

follow up: find particular sol'n given $y(1)=3$, $y'(1)=2$, $y''(1)=1$

tip: use Wronskian instead of taking derivatives again

$$y(x) = c_1 x + c_2 x\ln x + c_3 x^2 \longrightarrow y(1) = c_1 + 0 + c_3 = 3$$
$$y'(x) = c_1 + c_2(\ln x + 1) + 2c_3 x \longrightarrow y'(1) = c_1 + c_2 + 2c_3 = 2$$
$$y''(x) = 0 + c_2\left(\frac{1}{x}\right) + 2c_3 \longrightarrow y''(1) = c_2 + 2c_3 = 1$$

show yourself later that $c_1 = 1$, $c_2 = -3$, $c_3 = 2$

$\therefore$ particular sol'n is $y(x) = x - 3x\ln x + 2x^2$

expand def'n from previous §:

thm: let $y_1, y_2, \ldots, y_n$ be n linearly independent sol'ns of a homogeneous eq'n $y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = 0$

n^{th} derivative

on open I where $p_i(x)$ are continuous

then there exists $c_1, c_2, \ldots, c_n$ s.t. $Y = c_1y_1 + c_2y_2 + \ldots c_ny_n$ is a general sol'n to the DE

thm: given $p_1, p_2, \ldots, p_n, f(x)$ are continuous on open I containing point a

then $y^{(n)} + p_1(x)y^{(n-1)} + \ldots = f(x)$ ← non-homogeneous

has a <u>unique</u> sol'n on I that satisfies initial conditions: $y(a) = b_0$, $y'(a) = b_1$, $\ldots$, $y^{(n-1)}(a) = b_{n-1}$

revisit $y''' + 3y'' + 4y' + 12y = 0$ w its linearly indep. sol'ns: $y_1 = e^{-3x}$, $y_2 = \cos 2x$, $y_3 = \sin 2x$

saw that $y(x) = c_1e^{-3x} + c_2\cos 2x + c_3\sin 2x$

considering aforementioned theorems:

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exists a particular solution that satisfies (for example)

let $a=0$ $\quad y(0) = b_0$

$y'(0) = b_1$

$y''(0) = b_2$

based on its Wronskian $W = \begin{vmatrix} e^{-3x} & \cos 2x & \sin 2x \\ -3e^{-3x} & -2\sin 2x & 2\cos 2x \\ 9e^{-3x} & -4\cos 2x & -4\sin 2x \end{vmatrix}$

setup for linear eqns are: $\begin{cases} y(0) = c_1 + c_2 + 0 = b_0 \\ y'(0) = -3c_1 + 0 + 2c_3 = b_1 \\ y''(0) = 9c_1 - 4c_2 + 0 = b_2 \end{cases}$

Next, consider the non-homogeneous scenario:

given a single, fixed $\underbrace{\text{particular sol'n}}_{y_p}$ of a non-homogeneous eq'n

↑ known

and Y is another sol'n

let $y_c = Y - y_p$ ⟵ LC

we'll see next time that y_c is a solution to the corresponding homogeneous eqn